DENSITY OF MILD MIXING PROPERTY FOR VERTICAL FLOWS OF ABELIAN DIFFERENTIALS

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Abstract. We prove that if $g \geq 2$ then the set of all Abelian differentials (M, ω) for which the vertical flow is mildly mixing is dense in every stratum of the moduli space $\mathcal{H}_g$. The proof is based on a sufficient condition in [3] for special flows over irrational rotations and under piecewise constant roof functions to be mildly mixing.

1. Abelian differentials and direction flows

For every natural $g \geq 2$ let $\mathcal{H}_g$ stand for the moduli space of equivalence classes of pairs (M, ω) where M is a compact Riemann surface of genus g and ω is a nonzero holomorphic 1-form on M (an Abelian differential). Two pairs (M, ω) and (M', ω') are identified if they are mapped to one another by a conformal homeomorphism. The space $\mathcal{H}_g$ is naturally stratified by the subsets $\mathcal{H}_g(m_1, \dots, m_\kappa)$ of Abelian differentials whose zeros have multiplicities $m_1, \dots, m_\kappa$. By the Euler-Poincaré formula $m_1 + \ldots + m_\kappa = 2g - 2$. Every stratum $\mathcal{H}_g(m_1, \dots, m_\kappa)$ is a complex-analytic orbifold of dimension $2g + \kappa - 1$. Moreover, $\mathcal{H}_g(m_1, \dots, m_\kappa)$ possesses a natural Lebesgue measure ν. Let us denote by $(\mathcal{U}_s)_{s\in\mathbb{R}}$ the periodic continuous flow on $\mathcal{H}_g$ defined by $\mathcal{U}_s(\omega) = e^{is}\omega$.

For every $\theta \in \mathbb{C}$ such that $|\theta| = 1$, the Abelian differential ω determines the direction field $v_\theta : M \to TM$ so that $\omega(v_\theta) = \theta$ for all points of M except the zeros of ω which are singular for v_θ. By the direction flow we will mean the flow $\mathcal{F}^\theta = \mathcal{F}^{\omega,\theta}$ generated by v_θ. The flows $\mathcal{F}^1$ and $\mathcal{F}^i$ are called horizontal and vertical respectively. Direction flows preserve the volume form $\frac{i}{2}\omega \wedge \overline{\omega}$ on M which vanishes only at zeros of ω. This form determines a finite volume measure μ_ω which is invariant for all direction flows.

A separatrix of $\mathcal{F}^\theta$ joining two singularities (not necessarily distinct) is called a saddle connection of $\mathcal{F}^\theta$. Recall that in every stratum for a.e. Abelian differential (M, ω) the vertical and the horizontal flows have no saddle connections.

We are interested in ergodic (mixing) properties of the vertical flow $\mathcal{F}^i$ for $g \geq 2$. Avila and Forni proved in [1] that for ν-almost all $(M, \omega) \in \mathcal{H}_g(m_1, \dots, m_\kappa)$ the vertical flow is weakly mixing with respect to the measure μ_ω. It follows from Katok's result in [5] that direction flows are never strongly mixing.

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In this paper we will restrict our attention to the mild mixing property for $\mathcal{F}^i$. A finite measure–preserving dynamical system is *mildly mixing* (see [4]) if its Cartesian product with an arbitrary ergodic conservative (finite or infinite) measure-preserving dynamical system remains ergodic. It is an immediate observation that the strong mixing of a dynamical system implies its mild mixing and mild mixing implies weak mixing. Recall that a measure-preserving flow $(T_t)_{t\in\mathbb{R}}$ on $(X,\mathcal{B},\mu)$ is *rigid* if there exists $t_n \to +\infty$ such that $\mu(T_{t_n}^{-1}A\triangle A) \to 0$ for all $A \in \mathcal{B}$. It was proved in [4] that a finite measure–preserving flow is mildly mixing if and only if it has no non-trivial rigid factors. Using the same methods as in the proof of Theorem 1.3 in [10], one can prove that for almost every $(M,\omega) \in \mathcal{H}_g(m_1,\ldots,m_\kappa)$ the vertical flow is rigid. It follows that the set $\mathcal{H}_{mm}$ of $(M,\omega) \in \mathcal{H}_g(m_1,\ldots,m_\kappa)$ for which the vertical flow is mildly mixing is of measure zero. Nevertheless, we prove that $\mathcal{H}_{mm}$ is dense in every stratum $\mathcal{H}_g(m_1,\ldots,m_\kappa)$ (see Theorem 12).

The proof of the density of $\mathcal{H}_{mm}$ is based on three components: a polygonal representation of Abelian differentials described in Section 2 where we follow [12], the Rauzy-Veech induction (Section 3) and a sufficient condition in [3] for special flows built over irrational rotations and under piecewise constant roof functions to be mildly mixing (see Proposition 8). The proof consists of two main steps. In the first step, using the Rauzy-Veech induction, we prove that a typical Abelian differential is approximated by Abelian differentials whose vertical flows are isomorphic to step special flows built over three intervals exchange transformations and under roof functions constant on the exchanged intervals (see Lemma 10). In the second step we apply the main result of [3]. It says that a special flow built over an irrational circle rotation by α and under a three steps roof function (with one jump at $1-\alpha$ and one jump at some point ξ) is mildly mixing for a dense set of the data (α, ξ and heights of the steps). Using the Rauzy-Veech induction again, it follows that the same result holds for step special flows over exchanges of three intervals, i.e. such special flows are mildly mixing for a dense set of data (see Corollary 9).

2. Interval exchange transformations and a construction of Abelian differentials

In this section we briefly describe a standard construction of Abelian differentials. For more details we refer the reader to [12] and [13].

2.1. Interval exchange transformations.

2.1. Interval exchange transformations. Let $\mathcal{A}$ be a d-element alphabet and let $\pi = (\pi_0,\pi_1)$ be a pair of bijections $\pi_\varepsilon : \mathcal{A} \to \{1,\ldots,d\}$ for $\varepsilon = 0,1$. We adopt the notation from [12]. The set of all such pairs we will denote by $\mathcal{P}_\mathcal{A}$. Denote by $\mathcal{P}^0_\mathcal{A}$ the subset of irreducible pairs, i.e. such that $\pi_1\circ\pi_0^{-1}\{1,\ldots,k\} \neq \{1,\ldots,k\}$ for $1 \le k < d$. Let $\mathcal{P}^*_\mathcal{A}$ stand for the set of irreducible pairs such that $\pi_1\circ\pi_0^{-1}(k+1) \neq \pi_1\circ\pi_0^{-1}(k)+1$ for $1 \le k < d$.

Let us consider $\lambda = (\lambda_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}_+^\mathcal{A}\setminus\{0\}$, where $\mathbb{R}_+ = [0,+\infty)$. Let

$$|\lambda| = \sum_{\alpha\in\mathcal{A}}\lambda_\alpha, \quad I = [0,|\lambda|) \text{ and } I_\alpha = \left[\sum_{\pi_0(\beta)<\pi_0(\alpha)}\lambda_\beta, \sum_{\pi_0(\beta)\le\pi_0(\alpha)}\lambda_\beta\right).$$

Then $|I_\alpha| = \lambda_\alpha$. Let Ω_π stand the matrix $[\Omega_{\alpha\,\beta}]_{\alpha,\beta\in\mathcal{A}}$ given by

$$\Omega_{\alpha\,\beta} = \begin{cases} +1 & \text{if } \pi_1(\alpha) > \pi_1(\beta) \text{ and } \pi_0(\alpha) < \pi_0(\beta) \\ -1 & \text{if } \pi_1(\alpha) < \pi_1(\beta) \text{ and } \pi_0(\alpha) > \pi_0(\beta) \\ 0 & \text{in all other cases.} \end{cases}$$

Given $(\lambda,\pi) \in \mathbb{R}^{\mathcal{A}}_+ \times \mathcal{P}^0_{\mathcal{A}}$ let $T_{(\lambda,\pi)} : [0,|\lambda|) \to [0,|\lambda|)$ stand for the *interval exchange transformation* (IET) on d intervals I_α, $\alpha \in \mathcal{A}$, which are rearranged according to the permutation π, i.e. $T_{(\pi,\lambda)}x = x + w_\alpha$ for $x \in I_\alpha$, where $w = \Omega_\pi\lambda$.

Definition. Let ∂I_α stand for the left end point of the interval I_α. A pair (λ,π) satisfies the *Keane condition* if $T^m_{(\lambda,\pi)}\partial I_\alpha \neq \partial I_\beta$ for all $m \geq 1$ and for all $\alpha,\beta \in \mathcal{A}$ with $\pi_0(\beta) \neq 1$.

It was proved by Keane in [6] that if $\pi \in \mathcal{P}^0_{\mathcal{A}}$ then for almost every λ the pair (λ,π) satisfies the Keane condition.

2.2. **Construction of Abelian differentials.** For each $\pi \in \mathcal{P}^0_{\mathcal{A}}$ denote by $\mathcal{T}^+_\pi$ the set of vectors $\tau = (\tau_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$ such that

$$\sum_{\pi_0(\alpha)\leq k} \tau_\alpha > 0 \text{ and } \sum_{\pi_1(\alpha)\leq k} \tau_\alpha < 0 \text{ for all } 1 \leq k < d. \tag{1}$$

Denote by $\mathcal{T}^+_{\pi,\lambda}$ the set of $\tau \in \mathcal{T}^+_\pi$ for which

$$\lambda_{\pi_\varepsilon^{-1}(k)} = \lambda_{\pi_\varepsilon^{-1}(k+1)} = 0 \Longrightarrow \tau_{\pi_\varepsilon^{-1}(k)} \cdot \tau_{\pi_\varepsilon^{-1}(k+1)} > 0 \text{ for } 1 \leq k < d,\ \varepsilon = 0,1. \tag{2}$$

Of course, $\mathcal{T}^+_\pi$ and $\mathcal{T}^+_{\pi,\lambda}$ are open convex cones.

Assume that $\tau \in \mathcal{T}^+_{\pi,\lambda}$ and set $\zeta_\alpha = \lambda_\alpha + i\tau_\alpha \in \mathbb{C}$ for each $\alpha \in \mathcal{A}$. Let $\Gamma(\pi,\lambda,\tau)$ stand for the closed curve on $\mathbb{C}$ formed by concatenation of vectors

$$\zeta_{\pi_0^{-1}(1)}, \zeta_{\pi_0^{-1}(2)}, \ldots, \zeta_{\pi_0^{-1}(d)}, -\zeta_{\pi_1^{-1}(d)}, -\zeta_{\pi_1^{-1}(d-1)}, \ldots, -\zeta_{\pi_1^{-1}(1)}$$

with starting point at zero. The curve $\Gamma(\pi,\lambda,\tau)$ determines a polygon $P(\pi,\lambda,\tau)$ on $\mathbb{C}$ with $2d$ sides which has d pairs of parallel sides with the same length. Condition (1) means that the first $d-1$ vertices of the polygon $\sum_{k=1}^{j} \zeta_{\pi_0^{-1}(k)}$, $j = 1,\ldots,d-1$ are on the upper half-plane and the last $d-1$ vertices $\sum_{k=1}^{j} \zeta_{\pi_1^{-1}(k)}$, $j = 1,\ldots,d-1$ are on the lower half-plane.

Definition. (see [12] and [15]) The *suspension surface* $M(\pi,\lambda,\tau)$ is a compact surface obtained by the identification of the sides of the polygon $P(\pi,\lambda,\tau)$ in each pair of parallel sides. The surface $M(\pi,\lambda,\tau)$ possesses a natural complex structure inherited from $\mathbb{C}$ and a holomorphic 1-form ω determined by the form dz. Therefore $M(\pi,\lambda,\tau)$ can be treated as an element of a moduli space.

The zeros of ω correspond to the vertices of the polygon $P(\pi,\lambda,\tau)$ and the vertical flow $\mathcal{F}^i$ moves up each point of $P(\pi,\lambda,\tau)$ vertically at the unit speed. Note that for every $s \in \mathbb{R}$, taking $\lambda_s + i\tau_s = e^{is}(\lambda + i\tau)$,

$$\text{if } \lambda_s \in \mathbb{R}^{\mathcal{A}}_+ \text{ and } \tau_s \in \mathcal{T}^+_\pi \text{ then } M(\pi,\lambda_s,\tau_s) = \mathcal{U}_s M(\pi,\lambda,\tau). \tag{3}$$

2.3. **Zippered rectangles and a special representation of the vertical flow.** Suspension surfaces can be defined in the terms of zippered rectangles introduced by Veech [9]. For every (π,λ,τ) with $\tau\in\mathcal{T}^+_{\pi,\lambda}$ let us consider the vector $h=h(\tau)=-\Omega_\pi\tau$. In view of (1), $h\in\mathbb{R}^{\mathcal{A}}_+$. Here the surface $M(\pi,\lambda,\tau)$ is obtained from the rectangles $I_\alpha\times[0,h_\alpha]$, $\alpha\in\mathcal{A}$ by an appropriate identification of parts of their sides. For example, the interval $I_\alpha\times\{h_\alpha\}$ is identified by a translation with $T_{(\pi,\lambda)}I_\alpha\times\{0\}$ for all $\alpha\in\mathcal{A}$ (see [9] for details).

In this representation the vertical flow $\mathcal{F}^i$ moves up each point of zippered rectangles vertically at the unit speed which yields the following fact.

Lemma 1. *If $\tau\in\mathcal{T}^+_{\pi,\lambda}$ then the vertical flow on $M(\pi,\lambda,\tau)$ has a special representation over the interval exchange transformation $T_{(\pi,\lambda)}$ and under the roof function*

$$f_h:I\to\mathbb{R}_+,\quad f_h=\sum_{\alpha\in\mathcal{A}}h_\alpha\chi_{I_\alpha},$$

i.e. the vertical flow and the special flow $T^{f_h}_{(\pi,\lambda)}$ are isomorphic as measure-preserving systems.

We will also need the following results.

Proposition 2 (see Proposition 3.30 in [13] or [11])**.** *If $m_i>0$ for $i=1,\ldots,\kappa$ then ν-almost every $(M,\omega)\in\mathcal{H}_g(m_1,\ldots,m_\kappa)$ may be represented in the form $M(\pi,\lambda,\tau)$, where $\#\mathcal{A}=2g+\kappa-1$.*

Remark 1. By the proof of Proposition 3.30 in [13], we can choose π from $\mathcal{P}^*_{\mathcal{A}}$.

Proposition 3 (see [9] and [11])**.** *For fixed π all Abelian differentials $M(\pi,\lambda,\tau)$ lie in the same stratum $\mathcal{H}_g(m_1,\ldots,m_\kappa)$ and the map*

$$\hat{\mathcal{H}}(\pi)=\{\pi\}\times(\mathbb{R}_+\setminus\{0\})^{\mathcal{A}}\times\mathcal{T}^+_\pi\ni(\pi,\lambda,\tau)\mapsto M(\pi,\lambda,\tau)\in\mathcal{H}_g(m_1,\ldots,m_\kappa)$$

is continuous.

3. Rauzy-Veech induction

In this section we describe the Rauzy-Veech induction renormalization procedure introduced for IETs by Rauzy in [8] and extended to zippered rectangles by Veech in [9].

Let $(\pi,\lambda)\in\mathcal{P}^0_{\mathcal{A}}\times(\mathbb{R}^{\mathcal{A}}_+\setminus\{0\})$ be a pair such that $\lambda_{\pi_0^{-1}(d)}\neq\lambda_{\pi_1^{-1}(d)}$. Set

$$\varepsilon(\lambda,\pi)=\begin{cases}0 & \text{if} \quad \lambda_{\pi_0^{-1}(d)}>\lambda_{\pi_1^{-1}(d)}\\ 1 & \text{if} \quad \lambda_{\pi_0^{-1}(d)}<\lambda_{\pi_1^{-1}(d)}.\end{cases}$$

We say that (π,λ) has type $\varepsilon(\lambda,\pi)$. For $\varepsilon=0,1$ let $R_\varepsilon:\mathcal{P}^0_{\mathcal{A}}\to\mathcal{P}^0_{\mathcal{A}}$ be defined by $R_\varepsilon(\pi_0,\pi_1)=(\pi_0',\pi_1')$, where

$$\begin{aligned}\pi'_\varepsilon(\alpha) &= \pi_\varepsilon(\alpha)\text{ for all }\alpha\in\mathcal{A}\text{ and}\\ \pi'_{1-\varepsilon}(\alpha) &= \begin{cases}\pi_{1-\varepsilon}(\alpha) & \text{if} \quad \pi_{1-\varepsilon}(\alpha)\le\pi_{1-\varepsilon}\circ\pi_\varepsilon^{-1}(d)\\ \pi_{1-\varepsilon}(\alpha)+1 & \text{if} \quad \pi_{1-\varepsilon}\circ\pi_\varepsilon^{-1}(d)<\pi_{1-\varepsilon}(\alpha)<d\\ \pi_{1-\varepsilon}\pi_\varepsilon^{-1}(d)+1 & \text{if} \quad \pi_{1-\varepsilon}(\alpha)=d.\end{cases}\end{aligned}$$

Moreover, let $\Theta_{\pi,\varepsilon}=[\Theta_{\alpha\,\beta}]_{\alpha,\beta\in\mathcal{A}}$ stand for the matrix

$$\Theta_{\alpha\,\beta}=\begin{cases}1 & \text{if }\alpha=\beta\\ 1 & \text{if }\alpha=\pi^{-1}_{1-\varepsilon}(d)\text{ and }\beta=\pi_\varepsilon^{-1}(d)\\ 0 & \text{in all other cases.}\end{cases}$$

The *Rauzy-Veech induction* of $T_{(\lambda,\pi)}$ is the first return map T' of $T_{(\lambda,\pi)}$ to the interval

$$\left[0, |\lambda| - \min(\lambda_{\pi_0^{-1}(d)}, \lambda_{\pi_1^{-1}(d)})\right).$$

As it was shown by Rauzy in [8], T' is also an IET on d-intervals, hence $T' = T_{(\lambda',\pi')}$ for some $(\lambda',\pi') \in \mathcal{P}^0_{\mathcal{A}} \times (\mathbb{R}^{\mathcal{A}}_+ \setminus \{0\})$. Moreover,

$$(\lambda',\pi') = (R_\varepsilon \pi, \Theta^{-1*}_{\pi,\varepsilon}\lambda), \text{ where } \varepsilon = \varepsilon(\pi,\lambda),$$

and B^* denotes the conjugate transpose of B. This renormalization procedure determines the transformation

$$\hat{R} : \mathcal{P}^0_{\mathcal{A}} \times \left(\mathbb{R}^{\mathcal{A}}_+ \setminus \{0\}\right) \to \mathcal{P}^0_{\mathcal{A}} \times \left(\mathbb{R}^{\mathcal{A}}_+ \setminus \{0\}\right), \quad \hat{R}(\pi,\lambda) = (R_{\varepsilon(\pi,\lambda)}\pi, \Theta^{-1*}_{\pi,\varepsilon(\pi,\lambda)}\lambda)$$

whenever $\lambda_{\pi_0^{-1}(d)} \neq \lambda_{\pi_1^{-1}(d)}$. Therefore the map $\hat{R}$ is well defined for all (π,λ) satisfying the Keane condition. Moreover, $\hat{R}(\pi,\lambda)$ fulfills the Keane condition for each such (π,λ). Consequently, $\hat{R}^n(\pi,\lambda)$ is well defined for all $n \geq 1$ and for all (π,λ) satisfying the Keane condition (see [14] for details).

3.1. **Rauzy graphs and Rauzy-Veech cocycle.**

Definition. Let us consider the relation $\sim$ on $\mathcal{P}^0_{\mathcal{A}}$ for which $\pi \sim \pi'$ if there exists $(\varepsilon_1,\ldots,\varepsilon_k) \in \{0,1\}^*$ such that $\pi' = R_{\varepsilon_k} \circ \ldots \circ R_{\varepsilon_1}\pi$. Then $\sim$ is an equivalence relation; its equivalence classes are called *Rauzy classes*.

Of course, for each Rauzy class $C \subset \mathcal{P}^0_{\mathcal{A}}$, the set $C \times \mathbb{R}^{\mathcal{A}}_+$ is $\hat{R}$–invariant.

Definition. A pair $\pi \in \mathcal{P}^0_{\mathcal{A}}$ is called *standard* if $\pi_1 \circ \pi_0^{-1}(1) = d$ and $\pi_1 \circ \pi_0^{-1}(d) = 1$.

Proposition 4 (see [8])**.** *Every Rauzy class contains a standard pair.*

Denote by $\Theta : C \times \mathbb{R}^{\mathcal{A}}_+ \to GL(d,\mathbb{Z})$ the *Rauzy-Veech cocycle*

$$\Theta(\pi,\lambda) = \Theta_{\pi,\varepsilon(\pi,\lambda)}.$$

If $(\pi',\lambda') = \hat{R}^n(\pi,\lambda)$ then $\lambda' = \Theta^{(n)}(\pi,\lambda)^{-1*}\lambda$, where

$$\Theta^{(n)}(\pi,\lambda) = \Theta(\hat{R}^{n-1}(\pi,\lambda)) \cdot \Theta(\hat{R}^{n-2}(\pi,\lambda)) \cdot \ldots \cdot \Theta(\hat{R}(\pi,\lambda)) \cdot \Theta(\pi,\lambda)$$

Remark 2. For every $\lambda \in (\mathbb{R}_+ \setminus \{0\})^{\mathcal{A}}$ we have $\varepsilon(\pi, \Theta^*_{\pi,\varepsilon}\lambda) = \varepsilon$. Indeed,

$$\begin{aligned}
\left(\Theta^*_{\pi,\varepsilon}\lambda\right)_{\pi_\varepsilon^{-1}(d)} &= \sum_{\alpha\in\mathcal{A}} (\Theta_{\pi,\varepsilon})_{\alpha\,\pi_\varepsilon^{-1}(d)}\lambda_\alpha = \lambda_{\pi_\varepsilon^{-1}(d)} + \lambda_{\pi_{1-\varepsilon}^{-1}(d)},\\
\left(\Theta^*_{\pi,\varepsilon}\lambda\right)_{\pi_{1-\varepsilon}^{-1}(d)} &= \sum_{\alpha\in\mathcal{A}} (\Theta_{\pi,\varepsilon})_{\alpha\,\pi_{1-\varepsilon}^{-1}(d)}\lambda_\alpha = \lambda_{\pi_{1-\varepsilon}^{-1}(d)},
\end{aligned}$$

and hence $\left(\Theta^*_{\pi,\varepsilon}\lambda\right)_{\pi_\varepsilon^{-1}(d)} > \left(\Theta^*_{\pi,\varepsilon}\lambda\right)_{\pi_{1-\varepsilon}^{-1}(d)}$. Therefore $\hat{R}(\pi, \Theta^*_{\pi,\varepsilon}\lambda) = (R_\varepsilon\pi, \lambda)$.

Now assume that $(\pi',\lambda') = \hat{R}(\pi,\lambda)$. Then for every $\lambda'' \in (\mathbb{R}_+ \setminus \{0\})^{\mathcal{A}}$,

$$\hat{R}(\pi, \Theta(\pi,\lambda)^*\lambda'') = \hat{R}(\pi, \Theta^*_{\pi,\varepsilon(\pi,\lambda)}\lambda'') = (R_{\varepsilon(\pi,\lambda)}, \lambda'') = (\pi',\lambda'').$$

It follows that for every (π,λ), $n \geq 1$ and $\lambda'' \in (\mathbb{R}_+ \setminus \{0\})^{\mathcal{A}}$

$$(\pi',\lambda') = \hat{R}^n(\pi,\lambda) \text{ implies } \hat{R}^n(\pi, \Theta^{(n)}(\pi,\lambda)^*\lambda'') = (\pi',\lambda'').$$

3.2. **Extended Rauzy-Veech induction.** For every Rauzy class $C\subset\mathcal{P}^0_{\mathcal{A}}$ let

$$\hat{\mathcal{H}}(C)=\{(\pi,\lambda,\tau):\pi\in C,\lambda\in\mathbb{R}^{\mathcal{A}}_+\setminus\{0\},\tau\in\mathcal{T}^+_\pi\}.$$

By the *extended Rauzy-Veech induction* we mean the map $\hat{\mathcal{R}}:\hat{\mathcal{H}}(C)\to\hat{\mathcal{H}}(C)$,

$$\hat{\mathcal{R}}(\pi,\lambda,\tau)=(R_{\varepsilon(\pi,\lambda)}\pi,\Theta^{-1*}_{\pi,\varepsilon(\pi,\lambda)}\lambda,\Theta^{-1*}_{\pi,\varepsilon(\pi,\lambda)}\tau)=(\hat{R}(\pi,\lambda),\Theta^{-1*}(\pi,\lambda)\tau).$$

By Lemma 18.1 in [12], if $(\pi',\lambda')=\hat{R}(\pi,\lambda)$ then $\Theta^{-1*}(\pi,\lambda)\tau\in\mathcal{T}^+_{\pi'}$, and hence $\hat{\mathcal{R}}:\hat{\mathcal{H}}(C)\to\hat{\mathcal{H}}(C)$ is well defined almost everywhere. Moreover, for every $n\geq 1$

$$\text{if } (\pi',\lambda',\tau')=\hat{\mathcal{R}}^n(\pi,\lambda,\tau) \text{ then } \lambda'=\Theta^{(n)}(\pi,\lambda)^{-1*}\lambda \text{ and } \tau'=\Theta^{(n)}(\pi,\lambda)^{-1*}\tau.$$

Lemma 5 (see e.g. Section 18 in [12])**.** *$M(\hat{\mathcal{R}}^n(\pi,\lambda,\tau))$ and $M(\pi,\lambda,\tau)$ are the same elements of the moduli space.*

Denote by $(\hat{\mathcal{T}}_s)_{s\in\mathbb{R}}$ the *Teichmüller flow* on $\hat{\mathcal{H}}(C)$,

$$\hat{\mathcal{T}}_s(\pi,\lambda,\tau)=(\pi,e^s\lambda,e^{-s}\tau).$$

The set $\mathcal{H}(C)=\{(\pi,\lambda,\tau)\in\hat{\mathcal{H}}(C):|\lambda|=1\}$ is a global cross-section for $(\hat{\mathcal{T}}_s)_{s\in\mathbb{R}}$. Let $t_R:\hat{\mathcal{H}}(C)\to\mathbb{R}_+$ be defined by

$$t_R(\pi,\lambda,\tau)=-\log\left(1-\lambda_{\pi^{-1}_{1-\varepsilon}(d)}/|\lambda|\right) \text{ whenever } (\pi,\lambda) \text{ has type } \varepsilon.$$

If $\hat{\mathcal{R}}(\pi,\lambda,\tau)=(\pi',\lambda',\tau')$ then $t_R(\pi,\lambda,\tau)=-\log(|\lambda'|/|\lambda|)$ and $|\lambda|=e^{t_R(\pi,\lambda,\tau)}|\lambda'|$. Let us consider the Rauzy-Veech renormalization map $\mathcal{R}:\mathcal{H}(C)\to\mathcal{H}(C)$ given by

$$\mathcal{R}=\hat{\mathcal{R}}\circ\hat{\mathcal{T}}_{t_R(\pi,\lambda,\tau)}(\pi,\lambda,\tau)=(\pi',\lambda'/|\lambda'|,\tau'|\lambda'|).$$

Let m stand for the restriction of the measure $d\pi\,d_1\lambda\,d\tau$ to the set $\mathcal{H}(C)$, where $d\pi$ is the counting measure on $\mathcal{P}^0_{\mathcal{A}}$, $d_1\lambda$ is the Lebesgue measure on

$$\Lambda_{\mathcal{A}}=\{\lambda\in(\mathbb{R}_+\setminus\{0\})^{\mathcal{A}}:|\lambda|=1\}$$

and $d\tau$ is the Lebesgue measure on $\mathbb{R}^{\mathcal{A}}$.

Theorem 6 (see Corollary 27.3 in [12])**.** *For every Rauzy class $C\subset\mathcal{P}^0_{\mathcal{A}}$ the measure m is an $\mathcal{R}$-invariant ergodic conservative measure on $\mathcal{H}(C)$.*

3.3. **Different special representations of the vertical flow.** Fix $(\pi,\lambda,\tau)\in\hat{\mathcal{H}}(C)$. Recall that the vertical flow $\mathcal{F}^i$ on $M(\pi,\lambda,\tau)$ has the special representation over $T_{(\pi,\lambda)}$ and under $f_h:I\to\mathbb{R}_+$, where $h=h(\pi,\lambda,\tau)=-\Omega_\pi\tau\in\mathbb{R}^{\mathcal{A}}_+$. Let $(\pi',\lambda',\tau')=\hat{\mathcal{R}}(\pi,\lambda,\tau)$ and $h'=-\Omega_{\pi'}\tau'$. In view of $\Omega_{\pi'}=\Theta(\pi,\lambda)\,\Omega_\pi\,\Theta^*(\pi,\lambda)$ (see Lemma 10.2 in [12]),

$$h'=-\Omega_{\pi'}\tau'=-\Omega_{\pi'}\,\Theta^{-1*}(\pi,\lambda)\tau=-\Theta(\pi,\lambda)\,\Omega_\pi\tau=\Theta(\pi,\lambda)h.$$

Since $M(\pi,\lambda,\tau)$ and $M(\pi',\lambda',\tau')$ are the same elements of the moduli space, the special flows $T^{f_h}_{(\pi,\lambda)}$ and $T^{f_{h'}}_{(\pi',\lambda')}$ are isomorphic. In fact, a more general result holds. We leave the proof of the following simple lemma to the reader.

Lemma 7. *For every interval exchange transformation $T_{(\pi,\lambda)}$ and $h\in\mathbb{R}^{\mathcal{A}}_+$ the special flows $T^{f_h}_{(\pi,\lambda)}$ and $T^{f_{\Theta(\pi,\lambda)h}}_{\hat{R}(\pi,\lambda)}$ are isomorphic.*

4. Special flows over irrational rotations and exchanges of three intervals

Let $A \subset \mathbb{R}$ be an additive subgroup. A collection of real numbers $x_1, \ldots, x_k$ is called independent over A if $a_1x_1 + \ldots + a_kx_k = 0$ for $a_1, \ldots, a_k \in A$ implies $a_1 = \ldots = a_k = 0$.

Remark 3. Let $T_\alpha : [0,1) \to [0,1)$ be an ergodic rotation $T_\alpha x = x + \alpha$. Since the set $\mathbb{Q} + \mathbb{Q}\alpha$ is countable, the set of all $(x_1, \ldots, x_k) \in \mathbb{R}^k$ such that $x_1, \ldots, x_k$ are independent over $\mathbb{Q} + \mathbb{Q}\alpha$ is G_δ and dense. Denote by DC_1 the set of irrational numbers $\alpha \in [0,1)$ which satisfy the following Diophantine condition: there exists $c > 0$ such that $|p - q\alpha| > c/q$ for all $p \in \mathbb{Z}$ and $q \in \mathbb{Z} \setminus \{0\}$. Since DC_1 is dense in $[0,1)$, the set

$$\mathfrak{M} = \{(\alpha, \xi) \in [0,1)^2 : \alpha \in DC_1, \xi \in (\mathbb{Q} + \mathbb{Q}\alpha) \setminus (\mathbb{Z} + \mathbb{Z}\alpha)\}$$

is dense in $[0,1)^2$.

Given $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$ a measure-preserving flow and $s > 0$, we denote by $\mathcal{S}^s$ the flow $(S_{st})_{t\in\mathbb{R}}$. As a consequence of Theorem 1.1 in [3] and Corollary 23 in [2] we obtain the following.

Proposition 8. *Let $(\alpha, \xi) \in \mathfrak{M}$ and let $a_1, a_2, a_3 \in \mathbb{R}$ be independent over $\mathbb{Q} + \mathbb{Q}\alpha$ and such that $f = a_1 + a_2\chi_{[0,\xi)} + a_3\chi_{[0,1-\alpha)} > 0$. Then the special flow built over T_α and under the roof function f is mildly mixing. Moreover, the flows T_α^f and $(T_\alpha^f)^s$ are not isomorphic for all positive $s \neq 1$.*

Let $\mathcal{A} = \{a, b, c\}$,

$$\pi_s = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}, \quad \pi_l = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \quad \pi_r = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix} \text{ and}$$

$$\Lambda_{\mathcal{A}}^l = \Lambda_{\mathcal{A}}^r = \Lambda_{\mathcal{A}}, \;\; \Lambda_{\mathcal{A}}^0 = \{\lambda \in \Lambda_{\mathcal{A}} : \lambda_a < \lambda_c\}, \;\; \Lambda_{\mathcal{A}}^1 = \{\lambda \in \Lambda_{\mathcal{A}} : \lambda_a > \lambda_c\}.$$

Let us consider four functions $\rho_\gamma : \Lambda_{\mathcal{A}}^\gamma \to [0,1]^2$, $\gamma \in \{l, r, 0, 1\}$ defined by

$$\rho_l(x_a, x_b, x_c) = (1 - x_a, 1 - x_c), \;\; \rho_r(x_a, x_b, x_c) = (x_c, x_a),$$

$$\rho_0(x_a, x_b, x_c) = \left(\frac{x_c - x_a}{1 - x_a}, \frac{x_a}{1 - x_a}\right), \;\; \rho_1(x_a, x_b, x_c) = \left(\frac{1 - x_a}{1 - x_c}, \frac{x_a}{1 - x_c}\right).$$

Obviously, $\rho_\gamma : \Lambda_{\mathcal{A}}^\gamma \to \rho(\Lambda_{\mathcal{A}}^\gamma)$ is a C^∞–diffeomorphism and $\rho_\gamma(\Lambda_{\mathcal{A}}^\gamma) \subset [0,1]^2$ is open for $\gamma = l, r, 0, 1$. Let

$$\gamma(\pi, \lambda) = \begin{cases} l & \text{if} \quad \pi = \pi_l \\ r & \text{if} \quad \pi = \pi_r \\ 0 & \text{if} \quad \pi = \pi_s \text{ and } \lambda \in \Lambda_{\mathcal{A}}^0 \\ 1 & \text{if} \quad \pi = \pi_s \text{ and } \lambda \in \Lambda_{\mathcal{A}}^1. \end{cases}$$

Let us consider $\rho : \mathcal{P}_{\mathcal{A}}^0 \times \Lambda_{\mathcal{A}} \to [0,1]^2$ given by $\rho(\pi, \lambda) = \rho_{\gamma(\pi,\lambda)}(\lambda)$. We will use the notation $(\alpha(\pi, \lambda), \xi(\pi, \lambda))$ for $\rho(\pi, \lambda)$.

Corollary 9. *For every $\pi \in \mathcal{P}_{\mathcal{A}}^0$, $\lambda \in \Lambda_{\mathcal{A}}$ and $h \in \mathbb{R}_+^{\mathcal{A}}$ if $\rho(\pi, \lambda) \in \mathfrak{M}$ and h_1, h_2, h_3 are independent over $\mathbb{Q} + \mathbb{Q}\alpha(\pi, \lambda)$ then the special flow $T_{(\pi,\lambda)}^{f_h}$ is mildly mixing.*

Proof. We will prove the claim for the cases r and 0. In the remaining cases the proof is similar, and we leave it to the reader.

Suppose that $\pi=\pi_r$, $\lambda\in\Lambda_{\mathcal{A}}$ and $h\in\mathbb{R}_+^{\mathcal{A}}$. Then $T_{(\pi,\lambda)}$ is isomorphic to the circle rotation by $\lambda_c=\alpha(\pi,\lambda)$ and

$$\begin{aligned} f_h &= h_a+(h_b-h_a)\chi_{[0,\lambda_a)}+(h_c-h_b)\chi_{[0,1-\lambda_c)} \\ &= h_a+(h_b-h_a)\chi_{[0,\xi(\pi,\lambda))}+(h_c-h_b)\chi_{[0,1-\alpha(\pi,\lambda))}. \end{aligned}$$

Suppose that $\rho(\pi,\lambda)=(\alpha(\pi,\lambda),\xi(\pi,\lambda))\in\mathfrak{M}$ and h_a,h_b,h_c are independent over $\mathbb{Q}+\mathbb{Q}\alpha(\pi,\lambda)$. Then h_a,h_b-h_a,h_c-h_b are independent over $\mathbb{Q}+\mathbb{Q}\alpha(\pi,\lambda)$. Now Proposition 8 implies the mild mixing of $T^{f_h}_{(\pi,\lambda)}$.

Next, suppose that $\pi=\pi_s$, $\lambda\in\Lambda^0_{\mathcal{A}}$, $h\in\mathbb{R}_+^{\mathcal{A}}$. Then (π,λ) has type 0. Let $(\pi',\lambda')=\hat{R}(\pi,\lambda)$ and $h'=\Theta(\pi,\lambda)h$. Thus $\pi'=\pi_r$,

$$\lambda'=\Theta(\pi,\lambda)^{-1*}\lambda=(\lambda_a,\lambda_b,\lambda_c-\lambda_a) \text{ and } h'=(h_a+h_c,h_b,h_c).$$

By Lemma 7, the special flows $T^{f_h}_{(\pi,\lambda)}$ and $T^{f_{h'}}_{(\pi_r,\lambda')}$ are isomorphic. Note that

$$\rho(\pi_r,\lambda'/|\lambda'|)=\rho_r(\lambda'/|\lambda'|)=\rho_r\left(\frac{\lambda_a}{1-\lambda_a},\frac{\lambda_b}{1-\lambda_a},\frac{\lambda_c-\lambda_a}{1-\lambda_a}\right)=\rho_0(\lambda)=\rho(\pi,\lambda).$$

Suppose that $\rho(\pi,\lambda)\in\mathfrak{M}$ and h_a,h_b,h_c are independent over $\mathbb{Q}+\mathbb{Q}\alpha(\pi,\lambda)$. It follows that h_a+h_c,h_b,h_c are independent over $\mathbb{Q}+\mathbb{Q}\alpha(\pi,\lambda)$. Since $\rho(\pi_r,\lambda'/|\lambda'|)=\rho(\pi,\lambda)$ and $\alpha(\pi_r,\lambda'/|\lambda'|)=\alpha(\pi,\lambda)$, we have $\rho(\pi_r,\lambda'/|\lambda'|)\in\mathfrak{M}$ and h'_a,h'_b,h'_c are independent over $\mathbb{Q}+\mathbb{Q}\alpha(\pi_r,\lambda'/|\lambda'|)$. By the first part of the proof, the special flow $T^{f_{h'}}_{(\pi_r,\lambda'/|\lambda'|)}$ is mildly mixing. It follows that $T^{f_{h'}}_{(\pi_r,\lambda')}$, and hence $T^{f_h}_{(\pi,\lambda)}$, is mildly mixing. □

5. Mild mixing of vertical flows

Let $\|x\|=\sum_{\alpha\in\mathcal{A}}|x_\alpha|$ for every $x\in\mathbb{R}^{\mathcal{A}}$. For every matrix $B=[b_{\alpha\,\beta}]_{\alpha\,\beta\in\mathcal{A}}$ with positive entries let $\nu(B)=\max_{\alpha,\beta,\gamma\in\mathcal{A}}b_{\alpha\,\beta}/b_{\alpha\,\gamma}$. Then

$$\left\|\frac{B\lambda'}{|B\lambda|}-\frac{B\lambda}{|B\lambda|}\right\|\leq\nu(B)^2\|\lambda-\lambda'\| \text{ for all } \lambda,\lambda'\in\Lambda_{\mathcal{A}}. \tag{4}$$

We will denote by $\operatorname{Arg}:\mathbb{C}\setminus\{0\}\to(-\pi,\pi]$ the principal argument function. Recall that for every z_1,z_2 with nonnegative real parts we have $\operatorname{Arg}(z_1+z_2)=\operatorname{Arg}z_1+\operatorname{Arg}z_2$ and $\operatorname{Arg}(\overline{z}_1)=-\operatorname{Arg}(z_1)$.

Let $\mathcal{A}=\{1,\ldots,d\}$, $d\geq 4$. Assume that $\overline{\pi}\in\mathcal{P}^0_{\mathcal{A}}$ is a standard pair such that $\bar{\pi}_0$ is the identity. Let

$$Z(\bar{\pi})=\{(\bar{\pi},\lambda,\tau):\lambda_1=\ldots=\lambda_{d-3}=0,(\bar{\pi},\lambda,\tau)\in\hat{\mathcal{H}}(C),\tau\in\mathcal{T}^+_{\pi,\lambda}\}.$$

Lemma 10. *The set $\{M(\bar{\pi},\lambda,\tau):(\bar{\pi},\lambda,\tau)\in Z(\bar{\pi})\}$ is dense in $M(\hat{\mathcal{H}}(C))$.*

Proof. The proof consists of four steps. In the first step, using the extended Veech-Rauzy induction, for almost every $(\pi,\lambda,\tau)\in\mathcal{H}(C)$ we find a representation of $M(\pi,\lambda,\tau)$ which is given by $(\bar{\pi},\lambda^{(n)},\tau^{(n)})=\hat{\mathcal{R}}^{k_n}(\pi,\lambda,\tau)$ so that the first $d-3$ sides of the polygon $P(\bar{\pi},\lambda^{(n)},\tau^{(n)})$ are almost parallel. In the second step, $(\bar{\pi},\lambda^{(n)},\tau^{(n)})$ is perturbed to get $(\bar{\pi},\lambda^{p\,(n)},\tau^{(n)})$ such that the first $d-3$ sides of the polygon $P(\bar{\pi},\lambda^{p\,(n)},\tau^{(n)})$ are parallel. To describe this perturbation we will need two auxiliary substeps passing by

$$(\tilde{\lambda}^{(n)},\tilde{\tau}^{(n)})=(\lambda^{(n)}/|\lambda^{(n)}|,\tau^{(n)}|\lambda^{(n)}|),\ (\tilde{\lambda}^{p\,(n)},\tilde{\tau}^{(n)})=(\lambda^{p\,(n)}/|\lambda^{(n)}|,\tau^{(n)}|\lambda^{(n)}|).$$

In the third step, $(\bar{\pi}, \lambda^{p\,(n)}, \tau^{(n)})$ is rotated by an angle θ_n ($\theta_n \to 0$ as $n \to \infty$) to obtain $(\bar{\pi}, \lambda^{r\,(n)}, \tau^{r\,(n)}) \in \hat{\mathcal{H}}(C)$ so that the first $d-3$ sides of the polygon $P(\bar{\pi}, \lambda^{r\,(n)}, \tau^{r\,(n)})$ are vertical, hence $(\bar{\pi}, \lambda^{r\,(n)}, \tau^{r\,(n)}) \in Z(\bar{\pi})$. In the final step, we show that $M(\bar{\pi}, \lambda^{r\,(n)}, \tau^{r\,(n)}) \to M(\pi, \lambda, \tau)$. In order to do this, applying the inverse of the renormalization, we prove that

$$(\pi, \lambda^{b\,(n)}, \tau) = \hat{\mathcal{R}}^{-k_n}(\bar{\pi}, \lambda^{p\,(n)}, \tau^{(n)}) \to (\pi, \lambda, \tau).$$

In view of (3), it follows that

$$M(\bar{\pi}, \lambda^{r\,(n)}, \tau^{r\,(n)}) = \mathcal{U}_{\theta_n} M(\bar{\pi}, \lambda^{p\,(n)}, \tau^{(n)}) = \mathcal{U}_{\theta_n} M(\pi, \lambda^{b\,(n)}, \tau) \to M(\pi, \lambda, \tau).$$

Step 1. Let A_n stand for the set of $(\bar{\pi}, \lambda, \tau) \in \mathcal{H}(C)$ such that

$$\lambda_j > 0, \quad \frac{\tau_1}{\lambda_1} > 1, \quad -\sum_{\bar{\pi}_1(k) \le j} \tau_k > \frac{\lambda_1}{\tau_1} \text{ for } j = 1, \ldots, d, \tag{5}$$

$$\left| \frac{\tau_1}{\lambda_1} - \frac{\tau_j}{\lambda_j} \right| < \frac{1}{n} \text{ for } j = 2, \ldots, d-3, \;\; \tau_j < 0 \text{ for } j = d-2, d-1, d, \tag{6}$$

$$\frac{\lambda_1}{\tau_1} \sum_{k=1}^{d-3} \tau_k + \lambda_{d-2} + \lambda_{d-1} < 1. \tag{7}$$

Note that if $(\bar{\pi}, \lambda, \tau) \in A_n$ then the first $d-3$ sides of the polygon $P(\bar{\pi}, \lambda, \tau)$ are almost parallel. Setting $\lambda = (1/d, \ldots, 1/d)$, $\tau_j = 2/d$ for $j = 1, \ldots, d-3$, $\tau_{d-2} = \tau_{d-1} = -1/2d$ and $\tau_d = -3$, since $\overline{\pi}$ is a standard pair, we get $(\bar{\pi}, \lambda, \tau) \in A_n$. It follows that A_n is a nonempty open subset of $\mathcal{H}(C)$ and hence $m(A_n) > 0$.

By Theorem 6, using standard Veech arguments (see [10, Ch. 3]), there exists $\Gamma > 0$ and a measurable subset $B \subset \mathcal{H}(C)$ such that $m(B^c) = 0$ and for every $(\pi, \lambda, \tau) \in B$ there exists a sequence $k_n \to +\infty$ such that $\mathcal{R}^{k_n}(\pi, \lambda, \tau) \in A_n$, $\Theta^{(k_n)}(\pi, \lambda)$ has positive entries and $\nu(\Theta^{(k_n)}(\pi, \lambda)^*) \le \Gamma$. Let $(\bar{\pi}, \lambda^{(n)}, \tau^{(n)}) = \hat{\mathcal{R}}^{k_n}(\pi, \lambda, \tau)$ and

$$(\bar{\pi}, \tilde{\lambda}^{(n)}, \tilde{\tau}^{(n)}) = \mathcal{R}^{k_n}(\pi, \lambda, \tau) = (\bar{\pi}, \lambda^{(n)}/|\lambda^{(n)}|, \tau^{(n)}|\lambda^{(n)}|).$$

Since $(\bar{\pi}, \tilde{\lambda}^{(n)}, \tilde{\tau}^{(n)}) \in A_n$, we have

$$\tilde{\lambda}^{(n)}_j > 0, \quad \frac{\tilde{\tau}^{(n)}_1}{\tilde{\lambda}^{(n)}_1} > 1, \quad -\sum_{\bar{\pi}_1(k) \le j} \tilde{\tau}^{(n)}_k > \frac{\tilde{\lambda}^{(n)}_1}{\tilde{\tau}^{(n)}_1} \text{ for } j = 1, \ldots, d, \tag{8}$$

$$\left| \frac{\tilde{\tau}^{(n)}_1}{\tilde{\lambda}^{(n)}_1} - \frac{\tilde{\tau}^{(n)}_j}{\tilde{\lambda}^{(n)}_j} \right| < \frac{1}{n} \text{ for } j = 2, \ldots, d-3, \;\; \tilde{\tau}^{(n)}_j < 0 \text{ for } j = d-2, d-1, d, \tag{9}$$

$$\frac{\tilde{\lambda}^{(n)}_1}{\tilde{\tau}^{(n)}_1} \sum_{k=1}^{d-3} \tilde{\tau}^{(n)}_k + \tilde{\lambda}^{(n)}_{d-2} + \tilde{\lambda}^{(n)}_{d-1} < 1. \tag{10}$$

Moreover, $\tau^{(n)} \in H^+_{\bar{\pi}}$. From (8) and (9), we have $\tilde{\tau}^{(n)}_j > 0$ for $j = 1, \ldots, d-3$.

Step 2. Let us consider $\tilde{\lambda}^{p\,(n)} \in \mathbb{R}^{\mathcal{A}}$ with

$$\tilde{\lambda}^{p\,(n)}_j = \begin{cases} \frac{\tilde{\lambda}^{(n)}_1}{\tilde{\tau}^{(n)}_1} \tilde{\tau}^{(n)}_j & \text{if} \quad j = 1, \ldots, d-3 \\ \tilde{\lambda}^{(n)}_j & \text{if} \quad j = d-2, d-1 \\ 1 - \sum_{j=1}^{d-1} \tilde{\lambda}^{p\,(n)}_j & \text{if} \quad j = d. \end{cases}$$

It follows from (10) that $\tilde{\lambda}^{p\,(n)} \in \Lambda_{\mathcal{A}}$. Since $\left|\frac{\tilde{\tau}_j^{(n)}}{\tilde{\lambda}_j^{p\,(n)}} - \frac{\tilde{\tau}_j^{(n)}}{\tilde{\lambda}_j^{(n)}}\right| < \frac{1}{n}$ and $\frac{\tilde{\lambda}_1^{(n)}}{\tilde{\tau}_1^{(n)}} < 1$, we obtain

$$|\tilde{\lambda}_j^{p\,(n)} - \tilde{\lambda}_j^{(n)}| < \frac{1}{n}\frac{\tilde{\lambda}_j^{p\,(n)}\tilde{\lambda}_j^{(n)}}{\tilde{\tau}_j^{(n)}} = \frac{\tilde{\lambda}_j^{(n)}}{n}\frac{\tilde{\lambda}_1^{(n)}}{\tilde{\tau}_1^{(n)}} < \frac{\tilde{\lambda}_j^{(n)}}{n} \text{ for } j = 1, \ldots, d-3,$$

and hence $|\tilde{\lambda}_d^{p\,(n)} - \tilde{\lambda}_d^{(n)}| < 1/n$. Therefore, $\|\tilde{\lambda}^{p\,(n)} - \tilde{\lambda}^{(n)}\| < 2/n$. Moreover, by (8),

$$-\frac{\sum_{\bar{\pi}_1(k)\leq j} \tilde{\tau}_k^{(n)}}{\sum_{\bar{\pi}_1(k)\leq j} \tilde{\lambda}_k^{p\,(n)}} > -\sum_{\bar{\pi}_1(k)\leq j} \tilde{\tau}_k^{(n)} > \frac{\tilde{\lambda}_1^{(n)}}{\tilde{\tau}_1^{(n)}} \text{ for } j = 1, \ldots, d. \tag{11}$$

Let $\lambda^{p\,(n)} = |\lambda^{(n)}|\tilde{\lambda}^{p\,(n)}$. As $\tau^{(n)} \in H_{\bar{\pi}}^+$, we have $(\bar{\pi}, \lambda^{p\,(n)}, \tau^{(n)}) \in \hat{\mathcal{H}}(C)$. Since $\tau^{(n)} = \tilde{\tau}^{(n)}/|\lambda^{(n)}|$, by (8), (9) and (11), we obtain

$$\frac{\tau_j^{(n)}}{\lambda_j^{p\,(n)}} = \frac{\tau_1^{(n)}}{\lambda_1^{(n)}} > \frac{1}{|\lambda^{(n)}|^2} \text{ for } j = 1, \ldots, d-3, \quad \tau_j^{(n)} < 0 \text{ for } j = d-2, d-1, d \tag{12}$$

and

$$\begin{aligned} -\frac{\sum_{\bar{\pi}_1(k)\leq j} \tau_k^{(n)}}{\sum_{\bar{\pi}_1(k)\leq j} \lambda_k^{p\,(n)}} &= -\frac{\sum_{\bar{\pi}_1(k)\leq j} \tilde{\tau}_k^{(n)}}{\sum_{\bar{\pi}_1(k)\leq j} \tilde{\lambda}_k^{p\,(n)}}\frac{1}{|\lambda^{(n)}|^2} \\ &> \frac{\tilde{\lambda}_1^{(n)}}{\tilde{\tau}_1^{(n)}}\frac{1}{|\lambda^{(n)}|^2} = \frac{\lambda_1^{(n)}}{\tau_1^{(n)}}\frac{1}{|\lambda^{(n)}|^4} > \frac{\lambda_1^{(n)}}{\tau_1^{(n)}} \end{aligned} \tag{13}$$

for $j = 1, \ldots, d$.

Step 3. Let

$$\theta_n = \pi/2 - \mathrm{Arg}(\lambda_1^{(n)} + i\tau_1^{(n)}) = \mathrm{Arg}(\tau_1^{(n)} + i\lambda_1^{(n)}) > 0.$$

Since $|\lambda^{(n)}| \to 0$, by (12), we obtain $\theta_n \to 0$. Let

$$\lambda^{r\,(n)} + i\tau^{r\,(n)} = e^{i\theta_n}(\lambda^{p\,(n)} + i\tau^{(n)}).$$

In this step we will prove that $(\bar{\pi}, \lambda^{r\,(n)}, \tau^{r\,(n)}) \in Z(\bar{\pi})$. As $\mathrm{Arg}(\lambda_j^{p\,(n)} + i\tau_j^{(n)}) = \mathrm{Arg}(\lambda_1^{(n)} + i\tau_1^{(n)})$ for $j = 1, \ldots, d-3$ and $-\pi/2 < \mathrm{Arg}(\lambda_j^{p\,(n)} + i\tau_j^{(n)}) < 0$ for $j = d-2, d-1, d$, we have

$$\mathrm{Arg}(\lambda_j^{r\,(n)} + i\tau_j^{r\,(n)}) = \mathrm{Arg}(\lambda_j^{p\,(n)} + i\tau_j^{(n)}) + \pi/2 - \mathrm{Arg}(\lambda_1^{(n)} + i\tau_1^{(n)}) = \pi/2$$

for $j = 1, \ldots, d-3$ and

$$-\pi/2 < \mathrm{Arg}(\lambda_j^{r\,(n)} + i\tau_j^{r\,(n)}) = \mathrm{Arg}(\lambda_j^{p\,(n)} + i\tau_j^{(n)}) + \pi/2 - \mathrm{Arg}(\lambda_1^{(n)} + i\tau_1^{(n)}) < \pi/2$$

for $j = d-2, d-1, d$. It follows that

$$\lambda_j^{r\,(n)} = 0,\ \tau_j^{r\,(n)} > 0 \text{ for } j = 1, \ldots, d-3 \text{ and } \lambda_j^{r\,(n)} > 0 \text{ for } j = d-2, d-1, d. \tag{14}$$

Since $\tau^{(n)} \in H_{\bar{\pi}}^+$, we have $0 < \mathrm{Arg}(\sum_{k=1}^{j} \lambda_k^{p\,(n)} + i\tau_k^{(n)}) < \pi/2$, and hence

$$\mathrm{Arg}(\sum_{k=1}^{j} \lambda_k^{r\,(n)} + i\tau_k^{r\,(n)}) = \mathrm{Arg}(\sum_{k=1}^{j} \lambda_k^{p\,(n)} + i\tau_k^{(n)}) + \theta_n > \mathrm{Arg}(\sum_{k=1}^{j} \lambda_k^{p\,(n)} + i\tau_k^{(n)}) > 0$$

for $j=1,\ldots,d-1$. Therefore, $\sum_{k=1}^{j}\tau_j^{r\,(n)}>0$ for $j=1,\ldots,d-1$. By (13),

$$\begin{aligned}0 &> \operatorname{Arg}(\sum_{\bar\pi_1(k)\le j}\lambda_k^{p\,(n)}+i\tau_k^{(n)})+\operatorname{Arg}(\tau_1^{(n)}+i\lambda_1^{(n)})\\ &= \operatorname{Arg}(\sum_{\bar\pi_1(k)\le j}\lambda_k^{p\,(n)}+i\tau_k^{(n)})+\theta_n=\operatorname{Arg}(\sum_{\bar\pi_1(k)\le j}\lambda_k^{r\,(n)}+i\tau_k^{r\,(n)})\end{aligned}$$

and hence $\sum_{\bar\pi_1(k)\le j}\tau_k^{r\,(n)}<0$ for all $j=1,\ldots,d$. Therefore, $(\bar\pi,\lambda^{r\,(n)},\tau^{r\,(n)})\in\hat{\mathcal{H}}(C)$. In view of (14), it follows that $(\bar\pi,\lambda^{r\,(n)},\tau^{r\,(n)})\in Z(\overline{\pi})$.

Step 4. Let

$$\lambda^{b\,(n)}=\Theta^{(k_n)}(\pi,\lambda)^*\lambda^{p\,(n)}=|\lambda^{(n)}|\Theta^{(k_n)}(\pi,\lambda)^*\tilde\lambda^{p\,(n)}=\frac{\Theta^{(k_n)}(\pi,\lambda)^*\tilde\lambda^{p\,(n)}}{|\Theta^{(k_n)}(\pi,\lambda)^*\tilde\lambda^{(n)}|}.$$

Since $\nu(\Theta^{(k_n)}(\pi,\lambda)^*)\le\Gamma$, by (4),

$$\|\lambda^{b\,(n)}-\lambda\|=\left\|\frac{\Theta^{(k_n)}(\pi,\lambda)^*\tilde\lambda^{p\,(n)}}{|\Theta^{(k_n)}(\pi,\lambda)^*\tilde\lambda^{(n)}|}-\frac{\Theta^{(k_n)}(\pi,\lambda)^*\tilde\lambda^{(n)}}{|\Theta^{(k_n)}(\pi,\lambda)^*\tilde\lambda^{(n)}|}\right\|\le\Gamma^2\|\tilde\lambda^{p\,(n)}-\tilde\lambda^{(n)}\|\le\frac{2\Gamma^2}{n}.$$

Moreover, by Remark 2,

$$\hat R^{k_n}(\pi,\lambda^{b\,(n)})=(\bar\pi,\Theta^{(k_n)}(\pi,\lambda)^{-1*}\lambda^{b\,(n)})=(\bar\pi,\lambda^{p\,(n)})$$

and

$$\hat{\mathcal{R}}^{k_n}(\pi,\lambda^{b\,(n)},\tau)=(\bar\pi,\Theta^{(k_n)}(\pi,\lambda)^{-1*}\lambda^{b\,(n)},\Theta^{(k_n)}(\pi,\lambda)^{-1*}\tau)=(\bar\pi,\lambda^{p\,(n)},\tau^{(n)}).$$

Hence $M(\bar\pi,\lambda^{p\,(n)},\tau^{(n)})=M(\pi,\lambda^{b\,(n)},\tau)$. In view of (3), it follows that

$$M(\bar\pi,\lambda^{r\,(n)},\tau^{r\,(n)})=\mathcal{U}_{\theta_n}M(\bar\pi,\lambda^{p\,(n)},\tau^{(n)})=\mathcal{U}_{\theta_n}M(\pi,\lambda^{b\,(n)},\tau).$$

Since $\|\lambda^{b\,(n)}-\lambda\|<2/n$ and $\theta_n\to 0$, by the continuity of the map M (see Proposition 3) and the flow $(\mathcal{U}_s)_{s\in\mathbb{R}}$, it follows that $M(\bar\pi,\lambda^{r\,(n)},\tau^{r\,(n)})\to M(\pi,\lambda,\tau)$ in the moduli space for every $(\pi,\lambda,\tau)\in B\subset\mathcal{H}(C)$. Furthermore, for every real $s>0$ we have $M(\bar\pi,s\lambda^{r\,(n)},\tau^{r\,(n)})\to M(\pi,s\lambda,\tau)$.

Let $\widetilde B=\{(\pi,s\lambda,\tau)\in\hat{\mathcal{H}}(C):(\pi,\lambda,\tau)\in B\}$. Since the topological support of m is $\mathcal{H}(C)$ and $m(B^c)=0$, the set B is dense in $\mathcal{H}(C)$, and hence $\widetilde B$ is dense in $\hat{\mathcal{H}}(C)$. As $(\bar\pi,s\lambda^{r\,(n)},\tau^{r\,(n)})\in Z(\overline{\pi})$, it follows that $M(Z(\overline{\pi}))$ is dense in $M(\hat{\mathcal{H}}(C))$. □

Lemma 11. *Suppose that $\mathcal{A}=\{1,\ldots,d\}$ with $d\ge 4$ and $\bar\pi\in\mathcal{P}^*_{\mathcal{A}}$ is a standard pair such that $\bar\pi_0=id$. Assume that $\tau_1,\ldots,\tau_d$ are independent over an additive subgroup $A\subset\mathbb{R}$. Let $h=-\Omega_{\overline{\pi}}\tau$. Then h_{d-2},h_{d-1},h_d are also independent over A.*

Proof. Suppose that $a_1h_{d-2}+a_2h_{d-1}+a_3h_d=0$ and $a_1,a_2,a_3\in A$. Since $\bar\pi\in\mathcal{P}^*_{\mathcal{A}}$, we have $\bar\pi_1(d-1)\ne\bar\pi_1(d-2)+1$. Hence there exists $1<s<d-1$ such that $\Omega_{s\,(d-2)}\ne\Omega_{s\,(d-1)}$. Since $\overline{\pi}$ is a standard pair,

$$\begin{aligned}h_{d-2}&=\tau_1+\ldots+\Omega_{s\,(d-2)}\tau_s+\ldots-\tau_d\\ h_{d-1}&=\tau_1+\ldots+\Omega_{s\,(d-1)}\tau_s+\ldots-\tau_d\\ h_d&=\tau_1+\ldots+\tau_s+\ldots+\tau_{d-1},\end{aligned}$$

and hence

$$(a_1+a_2+a_3)\tau_1+\ldots+(a_1\Omega_{s\,(d-2)}+a_2\Omega_{s\,(d-1)}+a_3)\tau_s+\ldots+(-a_1-a_2)\tau_d=0.$$

Therefore

$$a_1 + a_2 + a_3 = a_1\Omega_{s\,(d-2)} + a_2\Omega_{s\,(d-1)} + a_3 = a_1 + a_2 = 0.$$

Since $\Omega_{s\,(d-2)} \neq \Omega_{s\,(d-1)}$, it follows that $a_1 = a_2 = a_3 = 0$. $\square$

Theorem 12. *If $g \geq 2$ then for every stratum $\mathcal{H}_g(m_1, \ldots, m_\kappa)$ there exists a dense subset $\mathcal{H}_{mm} \subset \mathcal{H}_g(m_1, \ldots, m_\kappa)$ such that for every $(M, \omega) \in \mathcal{H}_{mm}$ its vertical flow is mildly mixing.*

Proof. By Proposition 2 and Remark 1, there exists a finite family $\mathcal{C}$ of Rauzy classes in $\mathcal{P}^*_{\mathcal{A}}$ ($\#\mathcal{A} = d = 2g + \kappa - 1 \geq 4$) such that $\bigcup_{C \in \mathcal{C}} M(\hat{\mathcal{H}}(C))$ is dense in $\mathcal{H}_g(m_1, \ldots, m_\kappa)$.

Let $\mathcal{A} = \{1, \ldots, d\}$. In view of Proposition 4 and Lemma 10, it suffices to show that for every $\bar\pi$ standard pair in C such that $\bar\pi_0 = id$ and for every $(\bar\pi, \lambda, \tau) \in Z(\bar\pi)$ there exists a sequence $\{(\bar\pi, \lambda^n, \tau^n)\}_{n\in\mathbb{N}}$ in $Z(\overline{\pi})$ such that $(\lambda^n, \tau^n) \to (\lambda, \tau)$ and the vertical flow for $M(\bar\pi, \lambda^n, \tau^n)$ is mildly mixing. Without loss of generality we can assume that $|\lambda| = 1$. Moreover, we can also assume that λ_{d-2}, λ_{d-1},λ_d are positive and $\lambda_{d-2} \neq \lambda_d$, because the set of all $(\bar\pi, \lambda, \tau) \in Z(\bar\pi)$ satisfying this condition is dense in $Z(\bar\pi)$.

Suppose that $(\bar\pi, \lambda, \tau)$ is an element of $Z(\bar\pi)$ such that λ_{d-2}, λ_{d-1},λ_d are positive and $\lambda_{d-2} \neq \lambda_d$. Let $h = h(\tau) = -\Omega_{\overline{\pi}}\tau$. Since $(\bar\pi, \lambda, \tau) \in Z(\bar\pi)$ and $\overline{\pi}$ is a standard pair, by Lemma 1, the vertical flow for $M(\bar\pi, \lambda, \tau)$ is isomorphic to the special flow $T^{f_{\tilde h}}_{(\tilde\pi,\tilde\lambda)}$, where $T_{(\tilde\pi,\tilde\lambda)}$ is an exchange on three intervals such that $\tilde\pi \in \mathcal{P}^0_{\{d-2,d-1,d\}}$ is equal to

$$\pi_r = \begin{pmatrix} d-2 & d-1 & d \\ d & d-2 & d-1 \end{pmatrix} \text{ or } \pi_s = \begin{pmatrix} d-2 & d-1 & d \\ d & d-1 & d-2 \end{pmatrix},$$

$\tilde\lambda = (\lambda_{d-2}, \lambda_{d-1}, \lambda_d) \in \Lambda_{\{d-2,d-1,d\}}$ and $f_{\tilde h}$ is determined by $\tilde h = (h_{d-2}, h_{d-1}, h_d)$. Let $\gamma = \gamma(\tilde\pi, \tilde\lambda)$. Since $\rho_\gamma : \Lambda^\gamma_{\{d-2,d-1,d\}} \to \rho_\gamma(\Lambda^\gamma_{\{d-2,d-1,d\}}) \subset [0,1]^2$ is a diffeomorphism and $\mathfrak{M}$ is dense in $[0,1]^2$, we can find a sequence $\{(\lambda^n_{d-2}, \lambda^n_{d-1}, \lambda^n_d)\}_{n\in\mathbb{N}}$ in $\Lambda^\gamma_{\{d-2,d-1,d\}}$ such that

$$(\lambda^n_{d-2}, \lambda^n_{d-1}, \lambda^n_d) \to \tilde\lambda \text{ and } \rho(\tilde\pi, (\lambda^n_{d-2}, \lambda^n_{d-1}, \lambda^n_d)) = \rho_\gamma(\lambda^n_{d-2}, \lambda^n_{d-1}, \lambda^n_d) \in \mathfrak{M}.$$

Setting $\lambda^n = (0, \ldots, 0, \lambda^n_{d-2}, \lambda^n_{d-1}, \lambda^n_d) \in \Lambda_{\mathcal{A}}$, we have $\widetilde{\lambda}^n = (\lambda^n_{d-2}, \lambda^n_{d-1}, \lambda^n_d)$ and $\lambda^n \to \lambda$. Since $\mathcal{T}^+_{\overline{\pi},\lambda^n}$ is open, there exists a sequence $\{\tau^n\}_{n\in\mathbb{N}}$ such that $\tau^n \in \mathcal{T}^+_{\overline{\pi},\lambda^n}$, $\tau^n \to \tau$ and $\tau^n_1, \ldots, \tau^n_d$ are independent over $\mathbb{Q} + \mathbb{Q}\alpha(\tilde\pi, \widetilde{\lambda}^n)$. In view of Lemma 11, $h_{d-2}(\tau^n), h_{d-2}(\tau^n), h_d(\tau^n)$ are also independent over $\mathbb{Q}+\mathbb{Q}\alpha(\tilde\pi, \tilde\lambda^n)$. By Corollary 9, it follows that $T^{f_{\tilde h(\tau^n)}}_{(\tilde\pi,\tilde\lambda^n)}$ is mildly mixing. Consequently, the vertical flow of $M(\overline{\pi}, \lambda^n, \tau^n)$ is also mildly mixing. As $(\overline{\pi}, \lambda^n, \tau^n) \in Z(\overline{\pi})$ and $(\lambda^n, \tau^n) \to (\lambda, \tau)$, the theorem follows. $\square$

Corollary 13. *If $g \geq 2$ then the set of Abelian differentials in $\mathcal{H}_g$ for which the vertical flow is mildly mixing is dense in $\mathcal{H}_g$.*

6. Measure-theoretical equivalence of Abelian differentials and some orbits of the Teichmüller flow

Definition. Two Abelian differentials (M,ω) and (M',ω') are measure-theoretical isomorphic if there exists a measure-preserving invertible map $\psi:(M,\omega)\to(M',\omega')$ such that $\psi\circ\mathcal{F}_s^{\omega,\theta}=\mathcal{F}_s^{\omega',\theta}\circ\psi$ for every $\theta\in S^1$ and $s\in\mathbb{R}$.

For every stratum $\mathcal{H}_g(m_1,\ldots,m_\kappa)$ let $(\mathcal{T}_t)_{t\in\mathbb{R}}$ stand for the Teichmüller geodesic flow on $\mathcal{H}_g(m_1,\ldots,m_\kappa)$. As a consequence of results from previous sections we obtain the following.

Theorem 14. *If $g\geq 2$ then there exists a dense subset $\mathcal{H}'\subset\mathcal{H}_g(m_1,\ldots,m_\kappa)$ such that for every $(M,\omega)\in\mathcal{H}'$ the Abelian differentials (M,ω) and $\mathcal{T}_s(M,\omega)$ are not measure-theoretically equivalent for every real $s\neq 0$.*

Proof. By Proposition 8 and the proof of Theorem 12, there exists a dense subset $\mathcal{H}'\subset\mathcal{H}_g(m_1,\ldots,m_\kappa)$ such that for every $(M,\omega)\in\mathcal{H}'$ if $\mathcal{F}$ stands for its vertical flow then the flows $\mathcal{F}^t$ and $\mathcal{F}$ are not isomorphic for every positive $t\neq 1$. Moreover, every element of $\mathcal{H}'$ can be represented in the form $M(\pi,\lambda,\tau)$. Fix $(M,\omega)\in\mathcal{H}'$ and real $s\neq 0$. By $\tilde{\mathcal{F}}$ denote the vertical flow for $\mathcal{T}_s(M,\omega)$. Let $(\pi,\lambda,\tau)\in\hat{\mathcal{H}}(C)$ be a triple such that $M(\pi,\lambda,\tau)=(M,\omega)$. Then

$$\mathcal{T}_s(M,\omega)=\mathcal{T}_sM(\pi,\lambda,\tau)=M(\mathcal{T}_s(\pi,\lambda,\tau))=M(\pi,e^s\lambda,e^{-s}\tau).$$

It follows that $\tilde{\mathcal{F}}$ is isomorphic to the special flow $T_{(\pi,e^s\lambda)}^{f_{h(e^{-s}\tau)}}=T_{(\pi,e^s\lambda)}^{e^{-s}f_{h(\tau)}}$. Moreover, $T_{(\pi,e^s\lambda)}^{e^{-s}f_{h(\tau)}}$ is isomorphic to $\left(T_{(\pi,\lambda)}^{f_{h(\tau)}}\right)^{e^s}$ via the map $(x,y)\mapsto(e^{-s}x,e^sy)$. It follows that $\tilde{\mathcal{F}}$ is isomorphic to $\mathcal{F}^{e^s}$. Therefore $\tilde{\mathcal{F}}$ is not isomorphic to $\mathcal{F}$. □

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